C1 Notes (Edexcel)

Specification content

Unit C1 – Core Mathematics

The examination

The examination will consist of one 1½ hour paper. It will contain about ten questions of varying length. The mark allocations per question will be stated on the paper. All questions should be attempted.

Formulae which candidates are expected to know are given in the appendix to this unit and these will not appear in the booklet, *Mathematical Formulae including Statistical Formulae and Tables*, which will be provided for use with the paper. Questions will be set in SI units and other units in common usage.

For this unit, candidates may **not** have access to any calculating aids, including log tables and slide rules.

Preamble

Construction and presentation of rigorous mathematical arguments through appropriate use of precise statements and logical deduction, involving correct use of symbols and appropriate connecting language is required. Candidates are expected to exhibit correct understanding and use of mathematical language and grammar in respect of terms such as 'equals', 'identically equals', 'therefore', 'because', 'implies', 'is implied by', 'necessary', 'sufficient', and notation such as $\therefore$, $\Rightarrow$, $\Leftarrow$ and $\Leftrightarrow$.

SPECIFICATION	NOTES
1. Algebra and functions	
Laws of indices for all rational exponents.	The equivalence of $a^{m/n}$ and $\sqrt[n]{a^m}$ should be known.
Use and manipulation of surds.	Candidates should be able to rationalise denominators.
Quadratic functions and their graphs.	
The discriminant of a quadratic function.	
Completing the square. Solution of quadratic equations.	Solution of quadratic equations by factorisation, use of the formula and completing the square.
Simultaneous equations: analytical solution by substitution.	For example, where one equation is linear and one equation is quadratic
Solution of linear and quadratic inequalities.	For example, $ax + b > cx + d$, $px^2 + qx + r \geq 0$, $px^2 + qx + r < ax + b$.

Algebraic manipulation of polynomials, including expanding brackets and collecting like terms, factorisation.	Candidates should be able to use brackets. Factorisation of polynomials of degree n, $n \le 3$, eg $x^3 + 4x^2 + 3x$. The notation $f(x)$ may be used. (Use of the factor theorem is *not* required.)
Graphs of functions; sketching curves defined by simple equations. Geometrical interpretation of algebraic solution of equations. Use of intersection points of graphs of functions to solve equations.	Functions to include simple cubic functions and the reciprocal function $y = k/x$ with $x \ne 0$. Knowledge of the term asymptote is expected.
Knowledge of the effect of simple transformations on the graph of $y = f(x)$ as represented by $y = af(x)$, $y = f(x) + a$, $y = f(x + a)$, $y = f(ax)$.	Candidates should be able to apply one of these transformations to any of the above functions [quadratics, cubics, reciprocal] and sketch the resulting graph. Given the graph of any function $y = f(x)$ candidates should be able to sketch the graph resulting from one of these transformations.

2. Coordinate geometry in the (x, y) plane

Equation of a straight line, including the forms $y - y_1 = m(x - x_1)$ and $ax + by + c = 0$. Conditions for two straight lines to be parallel or perpendicular to each other.	To include (i) the equation of a line through two given points, (ii) the equation of a line parallel (or perpendicular) to a given line through a given point. For example, the line perpendicular to the line $3x + 4y = 18$ through the point (2, 3) has equation $y - 3 = \frac{4}{3}(x - 2)$.

3. Sequences and series

Sequences, including those given by a formula for the nth term and those generated by a simple relation of the form $x_{n+1} = f(x_n)$.	
Arithmetic series, including the formula for the sum of the first n natural numbers.	The general term and the sum to n terms of the series are required. The proof of the sum formula should be known. Understanding of Σ notation will be expected.

4. Differentiation

The derivative of $f(x)$ as the gradient of the tangent to the graph of $y = f(x)$ at a point; the gradient of the tangent as a limit; interpretation as a rate of change; second order derivatives.	For example, knowledge that $\frac{dy}{dx}$ is the rate of change of y with respect to x. Knowledge of the chain rule is not required. The notation $f'(x)$ may be used.
Differentiation of x^n, and related sums and differences.	E.g., for $n \neq 1$, the ability to differentiate expressions such as $(2x + 5)(x - 1)$ and $\frac{x^2 + 5x - 3}{3x^{1/2}}$ is expected.
Applications of differentiation to gradients, tangents and normals.	Use of differentiation to find equations of tangents and normals at specific points on a curve.

5. Integration

Indefinite integration as the reverse of differentiation.	Candidates should know that a constant of integration is required.
Integration of x^n.	For example, the ability to integrate expressions such as $\frac{1}{2}x^2 - 3x^{-\frac{1}{2}}$, $\frac{(x+2)^2}{x^{\frac{1}{2}}}$ is expected. Given $f'(x)$ and a point on the curve, candidates should be able to find an equation of the curve in the form $y = f(x)$.

Appendix C1 – Formulae

This appendix lists formulae that candidates are expected to remember and that may not be included in formulae booklets.

Quadratic equations

$ax^2 + bx + c = 0$ has roots $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Differentiation

function	derivative
x^n	nx^{n-1}

Integration

function	integral
x^n	$\dfrac{1}{n+1}x^{n+1} + c,\ n \neq -1$

Core Mathematics C1

Mensuration

Surface area of sphere = $4\pi r^2$

Area of curved surface of cone = $\pi r \times$ slant height

Arithmetic series

$u_n = a + (n - 1)d$

$S_n = \frac{1}{2}n(a + l) = \frac{1}{2}n[2a + (n - 1)d]$

Indices

Laws of indices for all rational exponents. The equivalence of $a^{\frac{m}{n}}$ *and* $\sqrt[n]{a^m}$ *should be known.*

We should already know from GCSE, the three Laws of indices :

(I)	$a^m \times a^n = a^{m+n}$	(e.g. $a^2 \times a^3 = a^5$)
(II)	$a^m \div a^n = a^{m-n}$	(e.g. $a^7 \div a^4 = a^3$)
(III)	$\left(a^m\right)^n = a^{mn}$	(e.g. $\left(a^5\right)^7 = a^{35}$)

In addition to these we need to remember the following:

REMEMBER $\sqrt{a} = a^{\frac{1}{2}}$, $\sqrt[n]{a} = a^{\frac{1}{n}}$, $\frac{1}{a} = a^{-1}$, $a^{-n} = \frac{1}{a^n}$ etc.

$$\sqrt{a} = a^{\frac{1}{2}},\quad \sqrt[n]{a} = a^{\frac{1}{n}}$$

So, for example:

$$64^{\frac{1}{2}} = \sqrt{64} = 8 \qquad 125^{\frac{1}{3}} = \sqrt[3]{125} = 5 \qquad 1024^{\frac{1}{10}} = \sqrt[10]{1024} = 2$$

$$\frac{1}{a} = a^{-1},\quad a^{-n} = \frac{1}{a^n}$$

So, for example:

$$2^{-3} = \frac{1}{2^3} = \frac{1}{8} \qquad 25^{-\frac{1}{2}} = \frac{1}{\sqrt{25}} = \frac{1}{5} \qquad 4^{-1} = \frac{1}{4}$$

$$a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$$

So, for example,

$$27^{\frac{4}{3}} = \left(27^{\frac{1}{3}}\right)^4 = 3^4 = 81 \qquad 4^{\frac{5}{2}} = \left(4^{\frac{1}{2}}\right)^5 = 2^5 = 32$$

$$625^{\frac{3}{4}} = \left(625^{\frac{1}{4}}\right)^3 = 5^3 = 125 \qquad 8^{-\frac{2}{3}} = \frac{1}{\left(8^{\frac{1}{3}}\right)^2} = \frac{1}{2^2} = \frac{1}{4}$$

REMEMBER $a^0 = 1$ and $a^1 = a$ for all values of a.

Exam Question 1

Given that $2^x = \frac{1}{\sqrt{2}}$ and $2^y = 4\sqrt{2}$,

(a) find the exact value of x and the exact value of y,

$\sqrt{2} = 2^{\frac{1}{2}}$ so $\frac{1}{\sqrt{2}} = 2^{-\frac{1}{2}}$. Hence $x = -\frac{1}{2}$

$4\sqrt{2} = 2^2 \times 2^{\frac{1}{2}} = 2^{\frac{5}{2}}$. Hence $y = \frac{5}{2}$

(b) calculate the exact value of 2^{y-x}.

$2^{y-x} = 2^{\frac{5}{2}-\frac{-1}{2}} = 2^3 = 8$

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Exam Question 2

(a) Given that $8 = 2^k$, write down the value of k.

$k = 3$

(b) Given that $4^x = 8^{2-x}$, find the value of x.

$4^x = 8^{2-x}$

$\Rightarrow \left(2^2\right)^x = \left(2^3\right)^{2-x}$

$\Rightarrow 2^{2x} = 2^{6-3x}$

$\Rightarrow 2x = 6 - 3x$

$\Rightarrow 5x = 6$

$\Rightarrow x = \frac{6}{5}$

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Surds

Use and manipulation of surds. Students should be able to rationalise denominators.

The square roots of certain numbers are integers (e.g. $\sqrt{9}=3$) but when this is not the case it is often easier to leave the square roots sign in the expression (e.g. it is simpler to write $\sqrt{3}$ than it is to write the value our calculator gives, i.e. 1.7320508…). Numbers of the form $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ etc. are called **surds**. We need to be able to simplify expressions involving surds.

It is important to realise the following : $\sqrt{ab}=\sqrt{a\times b}=\sqrt{a}\times\sqrt{b}$ $\quad e.g.\ \sqrt{6}=\sqrt{2\times 3}=\sqrt{2}\times\sqrt{3}$.

Now we see from the above that we can sometimes simplify surds.

For example

$$\begin{aligned}
&\sqrt{75}\\
&=\sqrt{25\times 3}\\
&=\sqrt{25}\times\sqrt{3}\\
&=5\times\sqrt{3}\\
&=5\sqrt{3}
\end{aligned}$$

or $\sqrt{50}=\sqrt{25\times 2}=5\sqrt{2}$

and $\dfrac{21}{\sqrt{3}}=\dfrac{21\sqrt{3}}{3}=7\sqrt{3}$

We can only simplify an expression of the form $\sqrt{a}$ if a has a factor which is a perfect square (as the number 25 was in the above example).

Multiplying surds: Multiply out brackets in usual way (using FOIL or a similar method). Then collect similar terms.

e.g. $\left(5-\sqrt{3}\right)\left(7+\sqrt{3}\right)=35+5\sqrt{3}-7\sqrt{3}-3=32-2\sqrt{3}$

$\left(3-\sqrt{5}\right)\left(3+\sqrt{5}\right)=9+3\sqrt{5}-3\sqrt{5}-5=4$

Dividing surds: For example simplify $\dfrac{\left(14+2\sqrt{5}\right)}{\left(3-\sqrt{5}\right)}$.

The denominator of this fraction is $\left(3-\sqrt{5}\right)$.

This is irrational and we need to be able to express this in a form in which the denominator is rational.

To do this we must multiply top and bottom of the fraction by an expression that will "rationalise the denominator".

We saw from above that $\left(3-\sqrt{5}\right)\left(3+\sqrt{5}\right)=4$ so we multiply top and bottom by $\left(3+\sqrt{5}\right)$.

So we have the following

$$\frac{\left(14+2\sqrt{5}\right)}{\left(3-\sqrt{5}\right)}=\frac{\left(14+2\sqrt{5}\right)\left(3+\sqrt{5}\right)}{\left(3-\sqrt{5}\right)\left(3+\sqrt{5}\right)}$$

$$=\frac{52+20\sqrt{5}}{4}$$

$$=13+4\sqrt{5}$$

Rationalising the denominator:

If the denominator is $a+\sqrt{b}$ then multiply top and bottom by $a-\sqrt{b}$.
If the denominator is $a-\sqrt{b}$ then multiply top and bottom by $a+\sqrt{b}$.
If the denominator is $\sqrt{a}+\sqrt{b}$ then multiply top and bottom by $\sqrt{a}-\sqrt{b}$.
If the denominator is $\sqrt{a}-\sqrt{b}$ then multiply top and bottom by $\sqrt{a}+\sqrt{b}$.

Exam Question

Given that $(2 + \sqrt{7})(4 - \sqrt{7}) = a + b\sqrt{7}$, where a and b are integers,

(a) find the value of a and the value of b.

$$\left(2+\sqrt{7}\right)\left(4-\sqrt{7}\right)=8+4\sqrt{7}-2\sqrt{7}-7=1+2\sqrt{7} \text{ so } a=1,\quad b=2.$$

Given that $\dfrac{2+\sqrt{7}}{4+\sqrt{7}} = c + d\sqrt{7}$ where c and d are rational numbers,

(b) find the value of c and the value of d.

$$\frac{2+\sqrt{7}}{4+\sqrt{7}}=\frac{\left(2+\sqrt{7}\right)\left(4-\sqrt{7}\right)}{\left(4+\sqrt{7}\right)\left(4-\sqrt{7}\right)}=\frac{1+2\sqrt{7}}{16-7}=\frac{1}{9}+\frac{2}{9}\sqrt{7}$$

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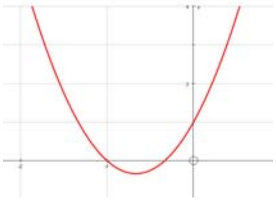
Quadratics

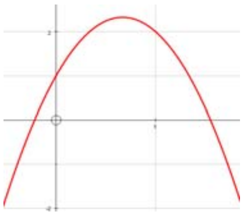
Quadratic functions and their graphs.

The graph of $y = ax^2 + bx + c$.

(i) $a > 0$



(ii) $a < 0$



The turning point can be determined by completing the square as we will see later.

The x-coordinates of the point(s) where the curve crosses the x-axis are determined by solving $ax^2 + bx + c = 0$.

The y-coordinates of the point where the curve crosses the y-axis is c. This is called the y-intercept.

The discriminant of a quadratic function.

In the quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, the **Discriminant** is the name given to $b^2 - 4ac$.

The value of the discriminant determines how many real solutions (or roots) there are.

If $b^2 - 4ac < 0$ then $ax^2 + bx + c = 0$ has no real solutions.
If $b^2 - 4ac = 0$ then $ax^2 + bx + c = 0$ has one real solution.
If $b^2 - 4ac > 0$ then $ax^2 + bx + c = 0$ has two real solutions.

Completing the square

Example 1 Complete the square on $x^2 + 6x + 11$.

We can write $x^2 + 6x = \left(x+3\right)^2 - 9$

and so we see that $x^2 + 6x + 11 = (x+3)^2 - 9 + 11 = (x+3)^2 + 2$.

"3" is obtained by halving "6"

So $\quad x^2 - 8x + 3 = (x-4)^2 - 16 + 3 = (x-4)^2 - 13$

and $\quad x^2 - x + 1 = (x - \tfrac{1}{2})^2 - \tfrac{1}{4} + \tfrac{4}{4} = (x - \tfrac{1}{2})^2 + \tfrac{3}{4}$.

We can use completing the square to find the turning points of quadratics.

Complete the square to find the turning point

Example : Find the turning point of $y = x^2 + 6x + 11$.

Completing the square on $y = x^2 + 6x + 11$ gives $y = x^2 + 6x + 11 = (x+3)^2 + 2$.

Since $(x+3)^2 \geq 0$ we have that $(x+3)^2 + 2 \geq 2$ and so $y \geq 2$.

It takes that minimum value of 2 when $(x+3)^2 = 0$, that is when $x = -3$.

Hence the minimum point at $\left(-3, 2\right)$.

Solution of quadratic equations. Solution of quadratic equations by factorisation, use of the formula and completing the square

There are three ways of solving quadratic equations

1. Factorising

e.g. Solve $x^2 - 7x + 12 = 0$.

Factorising gives $x^2 - 7x + 12 = (x-3)(x-4)$

So we have $(x-3)(x-4) = 0$ and so $x = 3$ or $x = 4$

e.g. Factorise $2x^2 + 7x + 6$:

(i) Find two numbers which add to give 7 and which multiply to give $2 \times 6 = 12$, i.e. 4 and 3

(ii) Write $2x^2 + 7x + 6 = 2x^2 + 4x + 3x + 6$

(iii) Factorise first two terms and second two terms to give $2x(x+2) + 3(x+2)$

(iv) Hence write as $(2x+3)(x+2)$

2. Completing the square.

e.g. Solve $x^2 - 10x + 19 = 0$

Completing the square gives $x^2 - 10x + 19 = (x-5)^2 - 25 + 19 = (x-5)^2 - 6$

So we have $(x-5)^2 - 6 = 0$.

Hence we see that $(x-5)^2 = 6$ and so $x - 5 = \pm\sqrt{6}$.

Thus we have solutions of $x = 5 \pm \sqrt{6}$

3. Quadratic Formula

e.g. Solve $x^2 - 10x + 19 = 0$ by the formula

We use the fact that the solutions to the equation $ax^2 + bx + c = 0$ are

$$x = \frac{-\mathrm{b} \pm \sqrt{\mathrm{b}^2 - 4\mathrm{ac}}}{2\mathrm{a}}.$$

In this case $a = 1$, $b = -10$ and $c = 19$ and so, using the formula, we have

$$x = \frac{10 \pm \sqrt{10^2 - 4 \times 1 \times 19}}{2} = \frac{10 \pm \sqrt{24}}{2}$$

Solving simple cubics

Example: Solve $x^3 - 5x^2 + 6x = 0$.

Each term has x in it so factorising leads to $x\left(x^2 - 5x + 6\right) = 0$.

Hence we have $x(x-2)(x-3) = 0$.

Thus the solutions are $x = 0$, $x = 2$ or $x = 3$.

Simultaneous Equations

Simultaneous equations: analytical solution by substitution. For example, where one equation is linear and one equation is quadratic

There are two methods for solving two linear simultaneous equations.

Method 1 – Cancellation

We cancel either the *x* or the *y* by multiplying the two equations by suitable numbers and then either adding or subtracting.

So, for example, solve $5x + 7y = 19$ (1)
and $3x + 2y = 7$ (2)

If we multiply (1) by 3 and (2) by 5 we get:

$$15x + 21y = 57 \qquad (1) \times 3$$
and
$$15x + 10y = 35 \qquad (2) \times 5$$

If we now subtract bottom from top we get $11y = 22$ and so $y = 2$. Substitute this value of *y* into (1) gives $5x + 14 = 19$ and so $x = 1$.

Method 2 – Substitution

We make either the *x* or the *y* the subject of one of the equations and then plug it back in to the other equation to find the other variable.

So, for example, solve $x + 2y = 13$ (1)
and $3x + 7y = 44$ (2)

From (1) we see that $x = 13 - 2y$.

If we now replace *x* in (2) with $x = 13 - 2y$ we get $3(13 - 2y) + 7y = 44$.

That leads to $39 - 6y + 7y = 44$ and so $y = 5$.

We also need to be able to solve simultaneous equations in which one of the equations is linear and one is quadratic .

So, for example, solve $3x + y = 10$ (1)
and $x^2 + 2xy + 2y^2 = 17$ (2)

We always use the method of substitution in these cases.

So, in our example, we use (1) to write $y = 10 - 3x$

We substitute this into (2) to give $x^2 + 2x(10 - 3x) + 2(10 - 3x)^2 = 17$.

Now expand brackets to give a quadratic equation

$$x^2 + 20x - 6x^2 + 2(100 - 60x + 9x^2) = 17$$

$$13x^2 - 100x + 200 = 17$$

$$13x^2 - 100x + 183 = 0$$

Factorise this to give $(x-3)(13x-61) = 0$ and so $x = 3$ or $x = \frac{61}{13}$.

We now use (1) to find y when $x = 3$ and we see that $y = 1$.
We also use (1) to find y when $x = \frac{61}{13}$ and we see that $y = -\frac{53}{13}$.

<u>It is important to give our solution as pairs</u>.

So we say that the solutions are $x = 3$ and $y = 1$ OR $x = \frac{61}{13}$ and $y = -\frac{53}{13}$.

Inequalities

Solution of linear and quadratic inequalities (For example $ax+b>cx+d$, $px^2+qx+r\geq 0$, $px^2+qx+r\geq ax+b$)

Linear Inequality
Remember that when you multiply or divide both sides by a negative number you must flip around the sign of the inequality.

For example:

$$4x+7\leq 7x-11$$
$$\Rightarrow -3x\leq -18$$
$$\Rightarrow x\geq 6$$

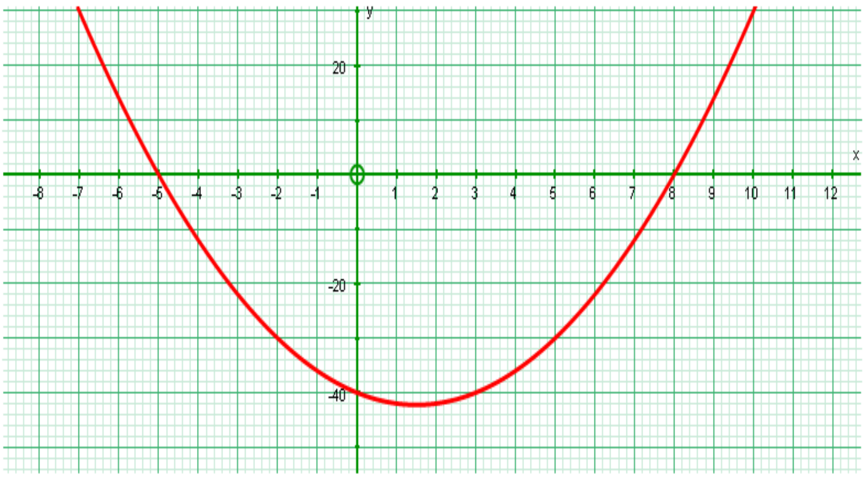
Quadratic inequality

Example 1 Solve $x^2-3x-40<0$.

We factorise to give $(x-8)(x+5)<0$.

The critical values are 8 and –5.

Think of the graph of $y=x^2-3x-40$.



We are looking below x-axis so we want <u>one</u> region, and hence <u>one</u> inequality.

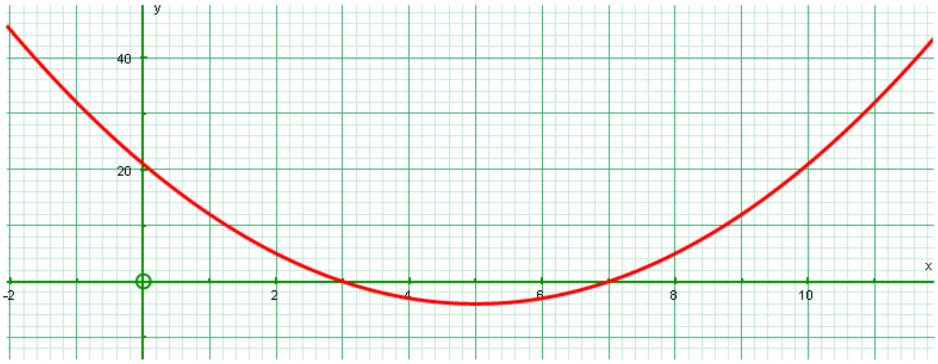
So the solution is $-5<x<8$.

Example 2 Solve $x^2 - 10x + 21 \geq 0$.

Factorise to give $(x-3)(x-7) \geq 0$.

Critical values are 3 and 7.

Think of the graph of $y = x^2 - 10x + 21$.



We are looking above x-axis so we want <u>two</u> regions, and hence <u>two</u> inequalities.

So the solution is $x \leq 3$ or $x \geq 7$.

How do you know if there are two regions or one?
Provided the x^2 term is positive…..

If the inequality sign involves **less than** ($\leq$ or $<$) then there is **one** region. (**One** is **less than** two).

If the inequality sign **involves more than** ($\geq$ or $>$) then there are **two** regions. (**Two** is **more than** one).

Algebraic Manipulation

Algebraic manipulation of polynomials, including expanding brackets and collecting like terms, factorisation. Students should be able to use brackets. Factorisation of polynomials of degree n, $n \le 3$ *eg* $x^3 + 4x^2 + 3x$.
The notation f*(x) may be used. (Use of the factor theorem is not required.)*

Expanding
When adding (or subtracting) two polynomials we simply add together (or subtract) the coefficients of the corresponding powers of *x*, for example $\left(x^2 - 2x + 7\right) + \left(3x^2 + 5x + 6\right) = 4x^2 + 3x + 13$

When multiplying two polynomials we simply multiply every term in one polynomial by every term in the other polynomial, for example

$$\begin{aligned}(x+4)\left(x^3+2x^2+5x+3\right) &= x\left(x^3+2x^2+5x+3\right)+4\left(x^3+2x^2+5x+3\right)\\ &= x^4+2x^3+5x^2+3x+4x^3+8x^2+20x+12 \;.\\ &= x^4+6x^3+13x^2+23x+12\end{aligned}$$

So for example we may be asked to find the x^2 term in $(x+1)\left(x^2-3x+5\right)$.

We do not need to multiply out the brackets completely. We see that there are two ways of getting an x^2 term. Either $x \times (-3x)$ or $1 \times \left(x^2\right)$. In total, then the x^2 term is $-2x^2$.

Factorising
When factorising a quadratic expression, such as $ax^2 + bx + c$ we are looking to write it in the form $(mx+n)(px+q)$. We need to consider two cases, when $a = 1$ and when $a \ne 1$.

Case 1: $a = 1$
For example, factorise $x^2 - x - 12$.

Step 1: Find two numbers whose product is -12 and whose sum is –1, i.e. –4 and 3.
Step 2: Express $x^2 - x - 12$ as $(x-4)(x+3)$.

Case 2: $a \ne 1$
For example, factorise $3x^2 - 14x + 8$.
Step 1: Find two numbers whose product is 24 (i.e. 3×8) and whose sum is –14, i.e. -12 and –2.
Step 2: Rewrite quadratic $3x^2 - 14x + 8 = 3x^2 - 12x - 2x + 8$.
Step 3: Factorise this as two separate linear expressions, so
$3x^2 - 12x - 2x + 8 = 3x(x-4) - 2(x-4)$.
Step 4: Note that there is a common linear factor and use this to give
$3x^2 - 14x + 8 = (3x-2)(x-4)$.

e.g. Factorise $x^2 + 7x + 10$:

(i) Find two numbers which add to give 7 and which multiply to give 10, i.e. 5 and 2

(ii) Write $x^2 + 7x + 10 = (x+5)(x+2)$

e.g. Factorise $2x^2 + 7x + 6$:

(i) Find two numbers which add to give 7 and which multiply to give $2 \times 6 = 12$, i.e. 4 and 3

(ii) Write $2x^2 + 7x + 6 = 2x^2 + 4x + 3x + 6$

(iii) Factorise first two terms and second two terms to give $2x(x+2) + 3(x+2)$

(iv) Hence write as $(2x+3)(x+2)$

Remember that $x^2 - 16 = (x-4)(x+4)$

Remember also that $x^2 - 5x = x(x-5)$

We may be asked to factorise a cubic but, for C1 there will be no constant term. So the expression may be $x^3 + 4x^2 + 3x$.

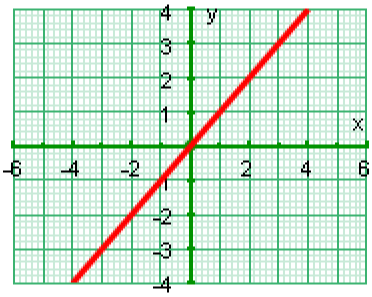
We see that there is a factor of *x* so we can write $x^3 + 4x^2 + 3x = x\left(x^2 + 4x + 3\right)$. We then need to factorise the quadratic as described above.

Graphs of Functions

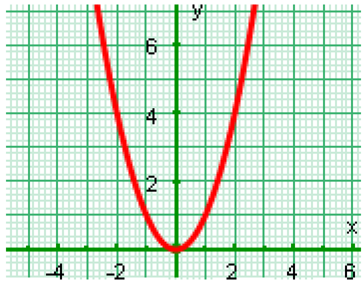
Graphs of functions; sketching curves defined by simple equations.

Here are some curves you should know.

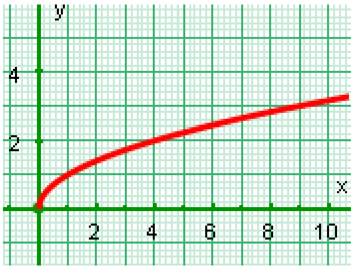
$y = x$



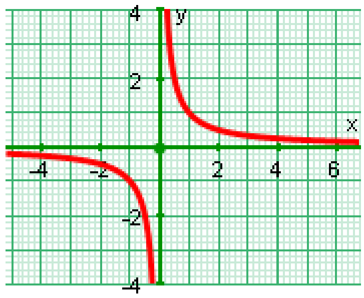
$y = x^2$



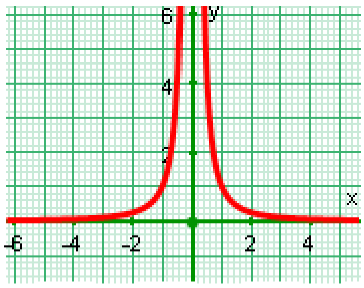
$y = \sqrt{x} = x^{\frac{1}{2}}$



$y = \dfrac{1}{x} = x^{-1}$



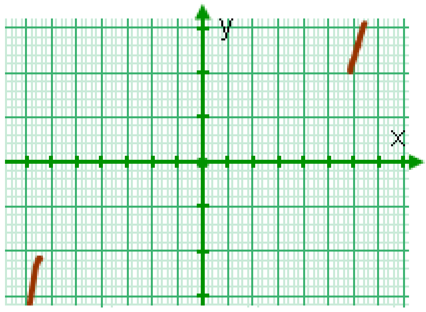
$y = \dfrac{1}{x^2} = x^{-2}$



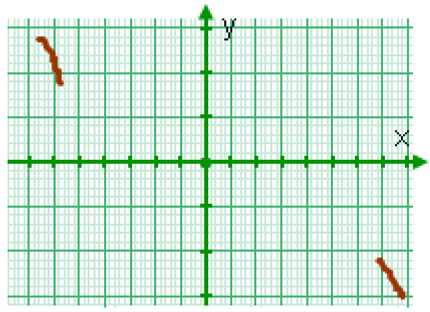
Geometrical interpretation of algebraic solution of equations. Use of intersection points of graphs of functions to solve equations. Functions to include simple cubic functions and the reciprocal function $y = k/x$ with $x \neq 0$.

Graphs of cubics - $y = ax^3 + bx^2 + cx + d$

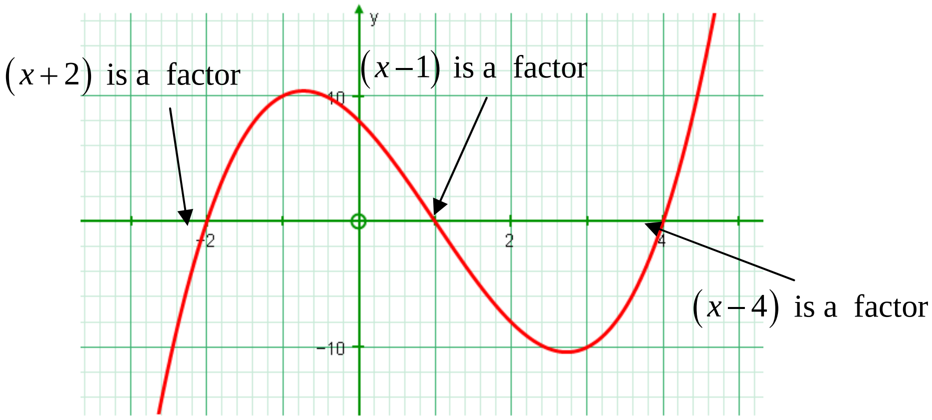
If $a > 0$ then curve goes from the bottom left to the top right:



If $a < 0$ then curve goes from the top left to the bottom right:



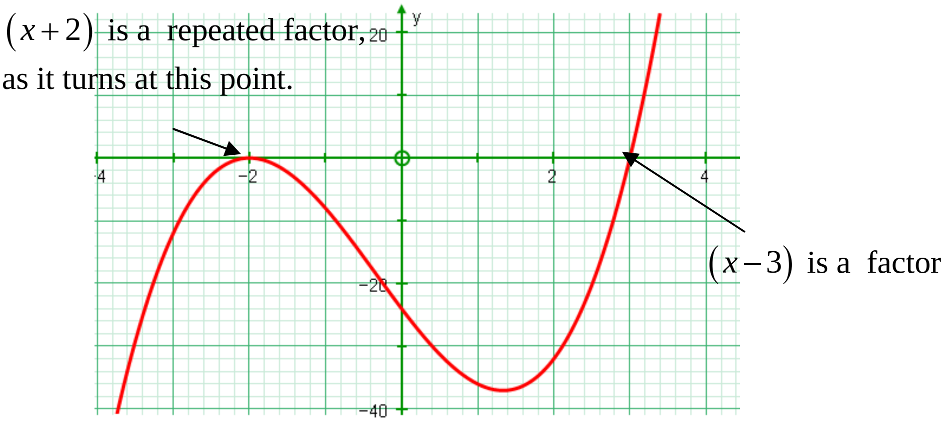
Three distinct linear factors…



So we see that $y = k(x+2)(x-1)(x-4)$.

The curve crosses the y axis at (0, 8). So $8 = k \times 2 \times (-1) \times (-4) = 8k$ and so $k = 1$.

One linear factor repeated…



So we see that $y = k\left(x+2\right)^2\left(x-3\right)$.

The curve crosses the *y* axis at (0, -24). So $8 = k\times\left(2\right)^2\times\left(-3\right) = -12k = -24$ and so $k = 2$.

Knowledge of the term asymptote is expected.

An asymptote is a line to which a given curve gets closer and closer. $y = \dfrac{1}{x}$ (as drawn on the previous page) has a vertical asymptote at $x = 0$ and a horizontal asymptote at $y = 0$.

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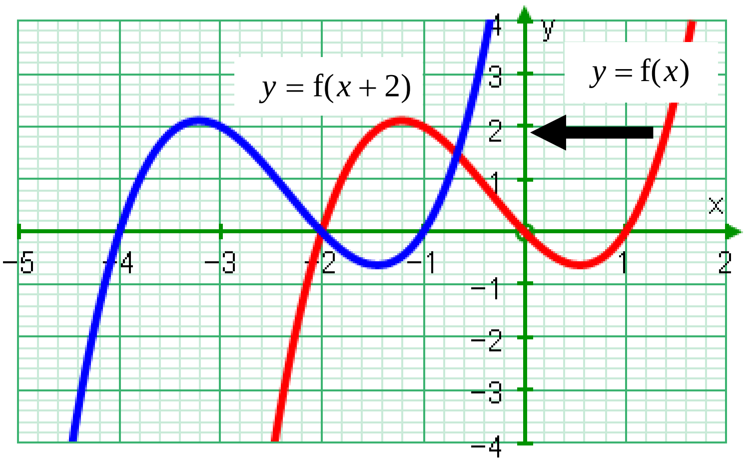
Transformations of Graphs

Knowledge of the effect of simple transformations on the graph of $y = \mathrm{f}(x)$ *as represented by* $y = a\,\mathrm{f}(x)$, $y = \mathrm{f}(x) + a$, $y = \mathrm{f}(x + a)$, $y = \mathrm{f}(ax)$. *Students should be able to apply one of these transformations to any of the above functions [quadratics, cubics, reciprocal] and sketch the resulting graph. Given the graph of any function y = f(x) students should be able to sketch the graph resulting from one of these transformations.*

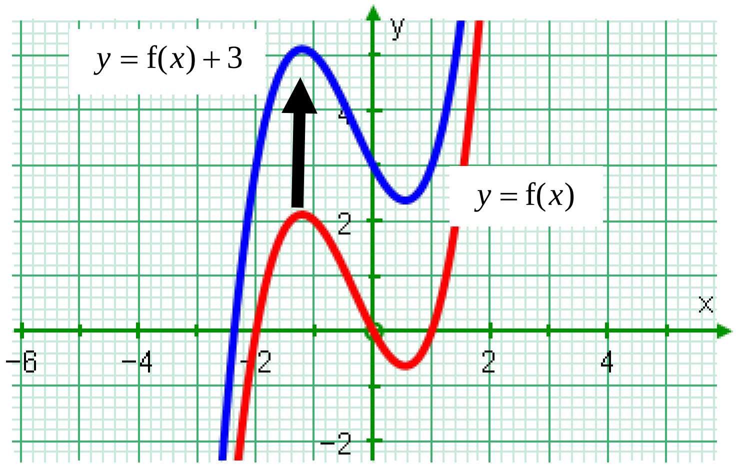
The following graphs are obtained as shown below:

$y = a\,\mathrm{f}(x)$	by stretching $y = \mathrm{f}(x)$ by a factor of a parallel to the y-axis.
$y = \mathrm{f}(x) + a$	by shifting $y = \mathrm{f}(x)$ by a in the positive y direction.
$y = \mathrm{f}(x + a)$	by shifting $y = \mathrm{f}(x)$ by a in the negative x direction.
$y = \mathrm{f}(ax)$	by stretching $y = \mathrm{f}(x)$ by a factor of $\frac{1}{a}$ parallel to the x-axis.

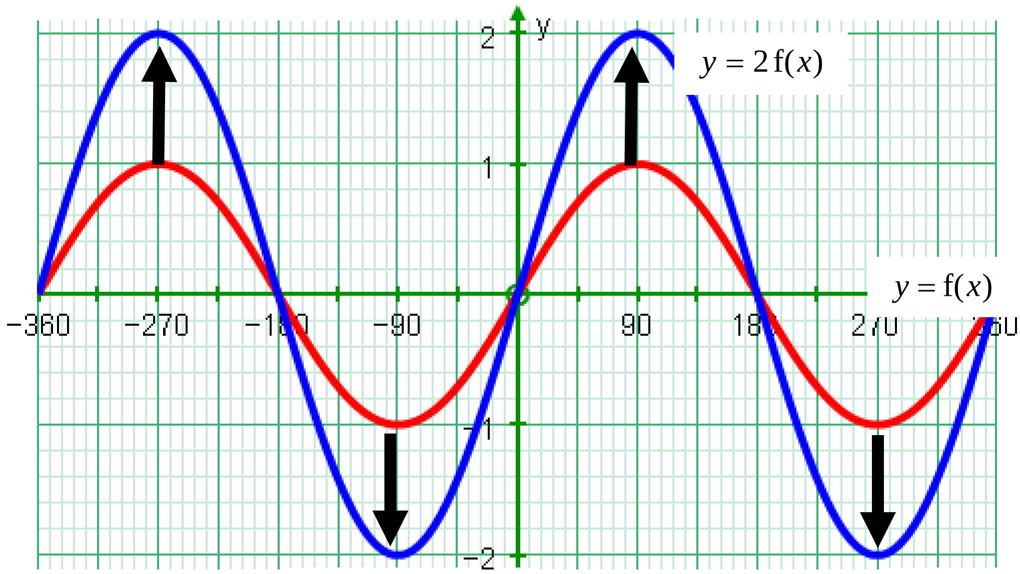
The connection between $y = \mathrm{f}(x)$ and $y = \mathrm{f}(x + 2)$ is shown below:



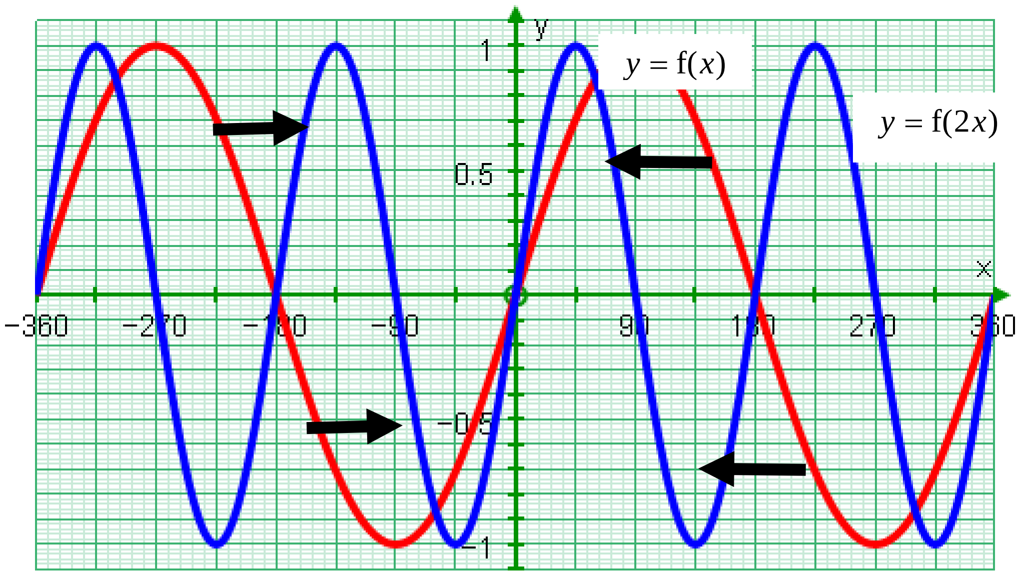
The connection between $y = \mathrm{f}(x)$ and $y = \mathrm{f}(x) + 3$ is shown below:



The connection between $y = \mathrm{f}(x)$ and $y = 2\,\mathrm{f}(x)$ is shown below:



The connection between $y = \mathrm{f}(x)$ and $y = \mathrm{f}(2x)$ is shown below:



Co-ordinate Geometry in the (*x*, *y*) plane

Equation of a straight line in forms $y = mx + c$, $y - y_1 = m(x - x_1)$ *and* $ax + by + c = 0$.
To include (i) the equation of a line through two given points, (ii) the equation of a line parallel (or perpendicular) to a given line through a given point. For example, the line perpendicular to the line $3x + 4y = 18$ *through the point (2, 3) has equation* $y - 3 = \frac{4}{3}(x - 2)$.

Conditions for two straight lines to be parallel or perpendicular to each other.

Equation of a straight line
From GCSE we know that $y = mx + c$ is the equation of a straight line where *m* is the gradient and *c* is the *y*-intercept.

We could use the form $y - y_1 = m(x - x_1)$. In this equation *m* is still the gradient and (x_1, y_1) is a point on the line

Both these equations can be written as $ax + by + c = 0$.

We can calculate the equation of a straight line if we know either (i) 2 points through which the line passes or (ii) one point and the gradient of the line.
In case (i) the first thing we do is to find the gradient and so we have now turned it into a problem of the type (ii) case.

Example 1
Find the equation of straight line which passes through $(3, 11)$ and $(-1, 27)$.

First of all find the gradient. Gradient $= \frac{27 - 11}{-1 - 3} = \frac{16}{-4} = -4$

Use the form $\frac{y - y_1}{x - x_1} = m$ to write the equation as $y - 11 = -4(x - 3)$ and then rearrange to give $y = -4x + 23$

Two straight lines are parallel if their gradients are equal.

Two straight lines are perpendicular if the product of their gradients is -1.

Example 2

Find the equation of straight line which passes through $(2,\ 5)$ and which has gradient $\frac{3}{4}$

We write $\frac{y-5}{x-2}=\frac{3}{4}$.

Cross multiplying gives $4(y-5)=3(x-2)$.
Hence $4y-20=3x-6$

So we have $3x-4y+14=0$

Example 3
Find the equation of straight line which passes through $(1,\ 4)$ and which is parallel to $2y+3x=7$.

The only line that are parallel to $2y+3x=7$ are lines of the form $2y+3x=c$.

We now simply "plug in" the point $(1,\ 4)$ to give $c=8+3=11$ so $2y+3x=11$.

Example 4
Find the equation of straight line which passes through $(-1,\ 3)$ and which is perpendicular to $4y+5x=11$.

Gradient of $4y+5x=11$ is $-5/4$ and so gradient of perpendicular is $4/5$.

It follows that the equation must be of the form $5y-4x=c$.

(NB: This can simply be obtained by swapping around the numbers in front of *x* and *y* and changing the sign of one of them).

We now "plug in" the point $(-1,\ 3)$ to give $c=15+4=19$ so $5y-4x=19$.

Sequences and Series

Sequences, including those given by a formula for the nth term and those generated by a simple relation of the form $x_{n+1} = \mathrm{f}(x_n)$.

There are sequences that be given in terms of *n* where *n* represents the number of the term. So for example we could have a sequence such as $x_n = n^2$. We can then see that the 10th term is 100 and so on.

We might have a sequence such as 1, 4, 7, 10,….
We can see that the *n*th term is $x_n = 3n - 2$. Alternatively we can see to find any term in the sequence we simply add 3 to the previous term. In this case we would write $x_{n+1} = x_n + 3$. When we write the sequence in this way we are said to be using an iterative formula.

To calculate terms in a sequence using an iterative formula we must be given a formula and a starting point. So in the above example we would have to write $x_1 = 1$, $x_{n+1} = x_n + 3$.

Example 1
Find the first four terms of the following sequence: $x_1 = 1$, $x_{n+1} = x_n{}^2 + 1$.

Plugging in $x_1 = 1$ gives $x_2 = 1^2 + 1 = 2$.
Plugging in $x_2 = 2$ gives $x_3 = 2^2 + 1 = 5$.
Plugging in $x_3 = 5$ gives $x_4 = 5^2 + 1 = 26$.

Example 2
Find the first four terms of the following sequence: $x_1 = 2$, $x_{n+1} = \dfrac{2}{x_n + 1}$.

Plugging in $x_1 = 1$ gives $x_2 = \dfrac{2}{2+1} = \dfrac{2}{3}$.

Plugging in $x_2 = \dfrac{2}{3}$ gives $x_3 = \dfrac{2}{1+\frac{2}{3}} = \dfrac{2}{\frac{5}{3}} = \dfrac{6}{5}$.

Plugging in $x_3 = \dfrac{6}{5}$ gives $x_4 = \dfrac{2}{1+\frac{6}{5}} = \dfrac{2}{\frac{11}{5}} = \dfrac{10}{11}$.

Arithmetic series, including the formula for the sum of the first n natural numbers.
The general term and the sum to n terms of the series are required. The proof of the sum formula should be known.

At GCSE we were asked to find the *n*th term of sequences such as 3, 8, 13, 18,…. We will now be looking at these sequences in more detail.

3, 8, 13, 18,…. is called an *arithmetic sequence* – that is a sequence in which the difference between consecutive terms is constant.

In an arithmetic sequence the first term is represented by *a* and the common difference is represented by *d*.

The first few terms of the sequence are $a, a+d, a+2d, a+3d, ...$

It follows from this that the *n*th term of this arithmetic sequence is $a+(n-1)d$.

We use u_n to represent the *n*th term of a sequence and so, in this example, we write $u_n = a+(n-1)d$.

So, for example, in the sequence 3, 8, 13, 18,…. we see that the first term is 3 and the common difference is 5. The *n*th term is, therefore, $3+(n-1)\times 5 = 3+5n-5 = 5n-2$.

In a similar way the *n*th term of the sequence $7, 10, 13, 16, ...$ is $u_n = 7+3(n-1) = 3n+4$.

Finding the Sum of Terms in an Arithmetic Sequence

For example how would we find the sum of the first 20 terms of the sequence 3, 8, 13, 18,…?

We have seen that *n*th term is $5n-2$ and so the 20th term is $5\times 20-2=98$.

So we want to find $S = 3+8+13+...+88+93+98$.

We could write the terms in reverse order to get $S = 98+93+88+...+13+8+3$

So we see that :

$$S = 98+93+88+...+13+8+3$$
$$S = 3+8+13+...+88+93+98$$

Adding these up in columns gives us $2S = 101+101+101+...+101+101+101$. There are 20 terms on the right hand side so we see that $2S = 101\times 20 = 2020$. Hence $S = 1010$.

In general we might want to find S where S is the sum of the first n terms of the sequence whose nth term is $u_n = a + (n-1)d$.

$$S = a \quad + (a+d) \quad + \ldots + (a + (n-2)d) + (a + (n-1)d)$$
$$S = (a + (n-1)d) + (a + (n-2)d) + \ldots + (a+d) \quad + a$$

Adding these two up gives us that:

$$2S = (2a + (n-1)d) + (2a + (n-1)d) + \ldots + (2a + (n-1)d) + (2a + (n-1)d)$$
$$= n(2a + (n-1)d)$$

So we have

$$\boxed{S = \frac{n}{2}(2a + (n-1)d)}$$

If we consider $1 + 2 + 3 + \ldots + n$ (that is the sum of the first n natural numbers) then the above formula for S gives us :

$$\boxed{1 + 2 + 3 + \ldots + n = \frac{n}{2}(2 + (n-1)) = \frac{n(n+1)}{2}}$$

Example 1

(a) Find the nth term of 7, 11, 15…
(b) Which term of 7, 11, 15…is equal to 51?
(c) Hence find $7 + 11 + 15 + \ldots\ldots + 51$.

(a) The first term, a, is 7 and the common difference, d, is 4.
The nth term is, therefore, $7 + (n-1) \times 4 = 4n + 3$.

(b) For what value of n is $4n + 3$ equal to 51?
$4n + 3 = 51$ gives us $4n = 48$ and so $n = 12$.

(c) We use $S = \frac{n}{2}(2a + (n-1)d)$ with $a = 7$, $d = 4$ and $n = 12$.
So we have $\frac{12}{2}(2 \times 7 + (12-1) \times 4) = 6 \times 58 = 348$.

Example 2

The sum of the first n terms of 3, 8, 13… is 1010. Find n.

The first term, a, is 3 and the common difference, d, is 5.

Using $S=\frac{n}{2}\left(2a+\left(n-1\right)d\right)$ with $a=3$ and $d=5$ we have that

$S=\frac{n}{2}\left(6+\left(n-1\right)\times 5\right)=\frac{n}{2}\left(5n+1\right)$.

We need to solve $\frac{n}{2}\left(5n+1\right)=1010$.

Multiplying both sides by 2 gives $n\left(5n+1\right)=2020$ and so $5n^2+n-2020=0$.

This factorises to give $\left(n-20\right)\left(5n+101\right)=0$ and so $n=20$.

Example 3

The tenth term of an arithmetic sequence is 67 and the sum of the first twenty terms is 1280. Find the common difference, d and the first term, a.

The 10$^{\text{th}}$ term is $a+9d$ (using $a+(n-1)d$) so we have $a+9d=67$.

The sum of the first twenty terms is $\frac{20}{2}\left(2a+19d\right)$ (using $S=\frac{n}{2}\left(2a+\left(n-1\right)d\right)$) so we have $10\left(2a+19d\right)=1280$. This can be rearranged to give $2a+19d=128$.

So we have two simultaneous equations. If we double both sides of $a+9d=67$ we get $2a+18d=134$. We now subtract this from $2a+19d=128$ to get $d=6$ and so $a=13$.

Understanding of Σ *notation will be expected.*

Know that $\sum_{r=1}^{n}u_r=u_1+u_2+u_3+...+u_n$, that $\sum_{r=1}^{n}r=1+2+3+...+n$ etc..

NB: $\sum_{r=1}^{n}1=1+1+1+...+1=n$

e.g. $\sum_{r=1}^{n}\left(3r+1\right)=4+7+...+\left(3n+1\right)$. Hence this is an arithmetic sequence with $a=4$ and $d=3$, so it follows that $\sum_{r=1}^{n}\left(3r+1\right)=\frac{n}{2}\left(8+3\left(n-1\right)\right)=\frac{n}{2}\left(3n+5\right)$.

Sigma Notation

Understanding of the $\sum$ *notation will be expected.*

Suppose we wanted to find $3+7+11+.....+79$.

It would make sense to have some sort of notation that enables us to write down 3+7+11+15+…..79 in a simpler form.

We already know that the *r*th term of the sequence is $4r-1$. So we want to find the sum of all terms of the form $4r-1$ from $r=1$ to $r=20$.

The notation we use to represent the sum is $\sum$. We then have to use a notation which indicates that we want the sum of terms of the form $4r-1$, that we start with $r=1$ and that we end with $r=20$.

The notation we use is $\sum_{r=1}^{20}(4r-1)$.

So $\sum_{r=1}^{20}(4r-1)=(4\times1-1)+(4\times2-1)+....+(4\times20-1)=3+7+11+.....+79$

So, for example,

$\sum_{r=1}^{n}(3r+1)=4+7+...+(3n+1)$. Hence this is an arithmetic sequence with $a=4$ and $d=3$, so it follows that $\sum_{r=1}^{n}(3r+1)=\frac{n}{2}(8+3(n-1))=\frac{n}{2}(3n+5)$.

NB: $\sum_{r=1}^{n}1=1+1+1+...+1=n$

Another example is $\sum_{r=1}^{n}2^r=2+2^2+2^3+...+2^n$.

Some more examples

$\sum_{r=1}^{4}r^2=1^2+2^2+3^2+4^2$ and $\sum_{r=3}^{5}r^2=3^2+4^2+5^2$

It is worth recognising that $\sum_{r=1}^{n}1=1+1+1+...+1=n$

So

The end value for *r*

The start value for *r*

$$\sum_{r=1}^{20}(4r-1)$$

The expression into which the values of *r* are put.

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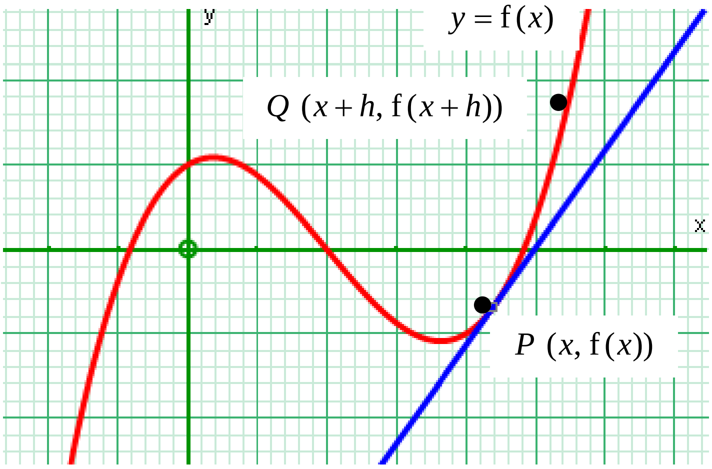
Differentiation

The derivative of f(*x*) *as the gradient of the tangent to the graph of* $y = \text{f}(x)$ *at a point; the gradient of the tangent as a limit; interpretation as a rate of change.*

For example, knowledge that $\frac{\text{d}y}{\text{d}x}$ *is the rate of change of y with respect to x. Knowledge of the chain rule is not required.*

Finding the gradient of the curve

Consider the curve shown below.



How do we find the gradient to the curve at the point *P*?

At GCSE we would have found the gradient by drawing a tangent to the curve at that point and then measuring the gradient of that tangent. This was only estimate – we now want to find the exact value of this gradient.

If *P* is a point on the curve $y = \text{f}(x)$ then the gradient of the curve at *P* is the gradient of the tangent to the curve at *P*.

If *Q* is another point on the curve then, as *Q* moves closer and closer to *P*, the gradient of the chord *PQ* get closer and closer to the gradient of the tangent to the curve at *P*.

The gradient of *PQ* is $\dfrac{\text{f}(x+h)-\text{f}(x)}{(x+h)-x} = \dfrac{\text{f}(x+h)-\text{f}(x)}{h}$.

The gradient of the curve at P is the gradient of the tangent to the curve at P. This is obtained by moving Q closer and closer to P (i.e. by making h tend to 0).

The gradient of the curve at P is denoted by $f'(x)$ or $\frac{dy}{dx}$.

So $f'(x) = \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \right)$ where "$\lim_{h \to 0}$" means that we make h tend to 0.

For example if $f(x) = x^2$ then $f(x+h) = (x+h)^2 = x^2 + 2hx + h^2$.

So we have that

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \right) \\ &= \lim_{h \to 0} \left(\frac{x^2 + 2hx + h^2 - x^2}{h} \right) \\ &= \lim_{h \to 0} \left(\frac{2hx + h^2}{h} \right) \\ &= \lim_{h \to 0} (2x + h) \\ &= 2x \end{aligned}$$

Thus we have proved that if $f(x) = x^2$ then $f'(x) = 2x$.

Alternatively we could write that if $y = x^2$ then $\frac{dy}{dx} = 2x$ or simply say "x^2 differentiates to $2x$".

This means that the gradient at any point on the curve $y = x^2$ is $2x$.

So, for example, at the point (3, 9) the gradient is 6.

Since $\frac{dy}{dx}$ is the gradient of the curve, we see that $\frac{dy}{dx}$ represents the rate of change of y with respect to x.

The notation f(*x*) *may be used. Differentiation of* x^n*, and related sums and differences.*
E.g., for $n \neq 1$*, the ability to differentiate expressions such as* $(2x+5)(x-1)$ *and* $\frac{x^2+5x-3}{x^{\frac{1}{2}}}$ *is expected.*
Applications of differentiation to gradients, tangents and normals.

Use of differentiation to find equations of tangents and normals at specific points on a curve.

We have just proved that if $y = x^2$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$.
We could have proved that if $y = x^3$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2$.

We already know that the gradient of $y = 1$ is 0 (it is a horizontal line) and that the gradient of the line $y = x$ is 1.

We could write this more formally by saying:

If $y = 1$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$. If $y = x$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = 1$

So we see the following:

y	$\frac{\mathrm{d}y}{\mathrm{d}x}$
1	0
x	1
x^2	$2x$
x^3	$3x^2$

We see a pattern here which continues. The pattern is that that if $y = x^n$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = nx^{n-1}$. This holds for all values of n – positive and negative, fractions and whole numbers.

So, for example if $y = x^6$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = 6x^5$.
The general result is as follows:

$$\boxed{\text{If } y = kx^n \text{ then } \frac{\mathrm{d}y}{\mathrm{d}x} = nkx^{n-1}.}$$

So, for example if $y = 2x^5$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = 10x^4$.

Example 1

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ when $y=(2x+5)(x-1)$

We first need to multiply out the bracket to get $y=2x^2+3x-5$.

We now differentiate using the rule on the previous page to get $\frac{\mathrm{d}y}{\mathrm{d}x}=4x+3$.

Example 2

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ when $y=\frac{x^2+5x-3}{x^{\frac{1}{2}}}$

We first need to rewrite *y* in the form $y=x^{\frac{3}{2}}+5x^{\frac{1}{2}}-3x^{\frac{1}{2}}$, using the laws of indices.

We now differentiate using the rule on the previous page to get $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{3}{2}x^{\frac{1}{2}}+\frac{5}{2}x^{-\frac{1}{2}}-\frac{3}{2}x^{-\frac{1}{2}}$.

Example 3

If $y=2x^2+3x+1$ then find the gradient of the curve at the point (2, 15).

First we find $\frac{\mathrm{d}y}{\mathrm{d}x}=4x+3$.

We then need to replace *x* in this expression with the *x* coordinate of the point we are looking at.

The *x* coordinate of the point (2, 15) is 2 so we substitute $x=2$ into the equation $\frac{\mathrm{d}y}{\mathrm{d}x}=4x+3$ to get $\frac{\mathrm{d}y}{\mathrm{d}x}=11$.

Some more examples...

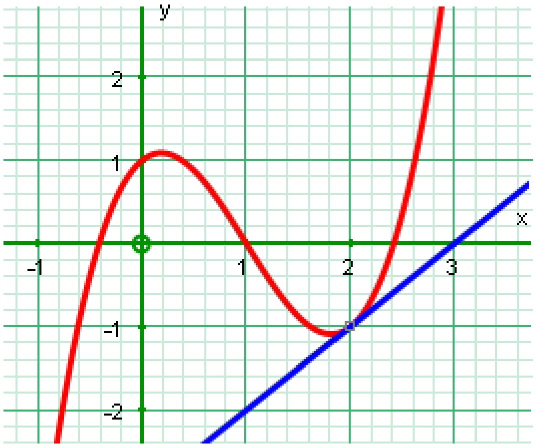
y	$\dfrac{dy}{dx}$
$(x+2)(x+3)=x^2+5x+6$	$2x+5$
$\dfrac{x^2+3}{x}=x+3x^{-1}$	$1-3x^{-2}$
$\dfrac{2x+6}{\sqrt{x}}=\dfrac{2x+6}{x^{\frac{1}{2}}}=2x^{\frac{1}{2}}+6x^{-\frac{1}{2}}$	$x^{-\frac{1}{2}}-3x^{-\frac{3}{2}}$
$\dfrac{(x+3)(x+4)}{x}=\dfrac{x^2+7x+12}{x}=x+7+12x^{-1}$	$1-12x^{-2}$
$\dfrac{(x+2)^2}{2\sqrt{x}}=\dfrac{x^2+4x+4}{2x^{\frac{1}{2}}}=\dfrac{1}{2}x^{\frac{3}{2}}+2x^{\frac{1}{2}}+2x^{-\frac{1}{2}}$	$\dfrac{3}{4}x^{\frac{1}{2}}+x^{-\frac{1}{2}}-x^{-\frac{3}{2}}$

Tangents

The gradient of a tangent to the curve at any point is the gradient of the curve at that point.

Example 4

Find the equation of the tangent to the curve $y = x^3 - 3x^2 + x + 1$ at the point (2, -1).



$\frac{dy}{dx} = 3x^2 - 6x + 1$.

We want to know $\frac{dy}{dx}$ at the point (2, -1) so we substitute $x = 2$ into the equation $\frac{dy}{dx} = 3x^2 - 6x + 1$ to get $\frac{dy}{dx} = 1$.

So the gradient of the tangent at (2, -1) is 1.

Using the earlier work on straight lines we know that the equation is $\frac{y+1}{x-2} = 1$.

Rearranging gives $y = x - 3$.

Normals

The gradient of a normal to the curve at any point is the <u>negative reciprocal</u> of the gradient of the curve at that point.

So..

if $\frac{dy}{dx}=3$ at a point on the curve, the gradient of the normal is $-\frac{1}{3}$.

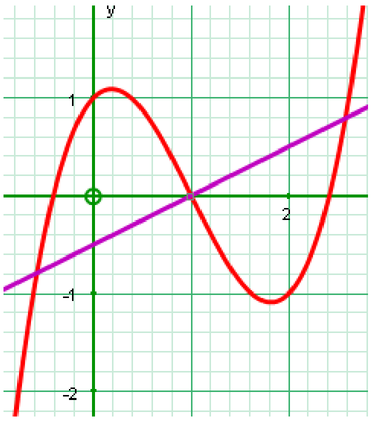
if $\frac{dy}{dx}=-4$ at a point on the curve, the gradient of the normal is $\frac{1}{4}$.

if $\frac{dy}{dx}=\frac{3}{5}$ at a point on the curve, the gradient of the normal is $-\frac{5}{3}$.

if $\frac{dy}{dx}=1$ at a point on the curve, the gradient of the normal is -1.

Example 5

Find the equation of the normal to the curve $y=x^3-3x^2+x+1$ at the point (1, 0).



$\frac{dy}{dx}=3x^2-6x+1$.

We want to know $\frac{dy}{dx}$ at the point (1, 0) so we substitute $x=1$ into the equation $\frac{dy}{dx}=3x^2-6x+1$ to get $\frac{dy}{dx}=-2$.

So the gradient of the normal at (1, 0) is $\frac{1}{2}$.

Using the earlier work on straight lines we know that the equation is $\frac{y-0}{x-1}=\frac{1}{2}$.

Rearranging gives $2y-x+1=0$.

Example 6

A curve C has equation $y = x^3 - 5x^2 + 5x + 2$.

(a) Find $\frac{dy}{dx}$ in terms of x.

The points P and Q lie on C. The gradient of C at both P and Q is 2. The x-coordinate of P is 3.

(b) Find the x-coordinate of Q.

(c) Find an equation for the tangent to C at P, giving your answer in the form $y = mx + c$, where m and c are constants.

(d) Find an equation for the normal to C at P, giving your answer in the form $ax + by + c = 0$, where m and c are constants.

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(a) $\frac{dy}{dx} = 3x^2 - 10x + 5$

(b) At P and Q , $\frac{dy}{dx} = 3x^2 - 10x + 5 = 2$ so $3x^2 - 10x + 3 = 0$

This gives $(3x-1)(x-3) = 0$ and so $x = 3$ or $x = \frac{1}{3}$. So x coordinate of Q is $\frac{1}{3}$.

(c) At P, $x = 3$, so $y = 3^3 - 5\times 3^2 + 5\times 3 + 2 = -1$. Also at P $\frac{dy}{dx} = 2$

Hence the equation is $\frac{y-(-1)}{x-3} = 2$. That is $y = 2x - 7$.

(d) At Q, $x = 3$, and $y = -1$. Also at P $\frac{dy}{dx} = 2$. So gradient of normal is $-\frac{1}{2}$.

Hence the equation is $\frac{y-(-1)}{x-3} = -\frac{1}{2}$. That is $2y + 2 = -x + 3$. Hence $2y + x - 1 = 0$.

Second order derivatives.

Consider the curve $y = x^3 - 3x^2 - 9x + 11$.

We see that $\frac{dy}{dx} = 3x^2 - 6x - 9$.

The “gradient of the gradient” is denoted by $\frac{d^2y}{dx^2}$. We see that in this example $\frac{d^2y}{dx^2} = 6x - 6$

Indefinite Integration

Indefinite integration as the reverse of differentiation. Students should be aware that a constant of integration is required.

Integration of x^n. *For example, the ability to integrate expressions such as* $\frac{1}{2}x^2 - 3x^{-\frac{1}{2}}$, $\frac{(x+2)^2}{x^{\frac{1}{2}}}$ *is expected.*
Given $\mathrm{f}'(x)$ *and a point on the curve, students should be able to find an equation of the curve in the form* $y = \mathrm{f}(x)$.

We know that if $y = 2x^3 + 5x$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = 6x^2 + 5$.

However we also that
if $y = 2x^3 + 5x + 2$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = 6x^2 + 5$.

or if $y = 2x^3 + 5x + 3$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = 6x^2 + 5$.

So if we were given $\frac{\mathrm{d}y}{\mathrm{d}x} = 6x^2 + 5$ and asked to find *y* we cannot get a definite answer.

As we have seen *y* could be $2x^3 + 5x$ or it could be $2x^3 + 5x + 2$ or it could be $2x^3 + 5x + 3$.

All we can say is that if $\frac{\mathrm{d}y}{\mathrm{d}x} = 6x^2 + 5$ then $y = 2x^3 + 5x + c$ where *c* can be any constant.

To find a particular solution of $\frac{\mathrm{d}y}{\mathrm{d}x} = 6x^2 + 5$ we need to have more information.

Example 1
If $\frac{\mathrm{d}y}{\mathrm{d}x} = 6x^2 + 5$ <u>and</u> the curve passes through the point (1, 13) then find *y*.

We see that $y = 2x^3 + 5x + c$
Since the curve passes through the point (1, 13), we have that $13 = 2\times1^2 + 5\times1 + c$ and so $c = 6$.

Hence see that the solution is $y = 2x^3 + 5x + 6$.

Notation

The "indefinite integral" of $f(x)$ is denoted by $\int f(x)\ dx$.

The general rule we use is $\int kx^p\ dx = k\dfrac{x^{p+1}}{p+1}+c\ \ (p \neq -1)$

Here are some examples

$\int 3\ dx = 3x + c$

$\int 5x\ dx = \dfrac{5}{2}x^2 + c$

$\int 4\sqrt{x}\ dx = \int 4x^{\frac{1}{2}}\ dx = \frac{8}{3}x^{\frac{3}{2}} + c$

$\int \dfrac{6}{x^2}\ dx = \int 6x^{-2}\ dx = -6x^{-1} + c$

NB We can only integrate an expression made up of terms of the form kx^p.

Example 2

Find $\int (2x+1)(2x+3)\ dx$.

We first of all need to express $(2x+1)(2x+3)$ as $4x^2+8x+3$

We now use the above rule to see that

$\int (2x+1)(2x+3)\ dx$

$= \int 4x^2+8x+3\ dx$

$= \dfrac{4}{3}x^3+4x^2+3x+c$

Example 3

Find $\int \dfrac{8x+x^2}{\sqrt{x}}\ dx$.

We first of all need to express $\dfrac{8x+x^2}{\sqrt{x}}$ as $\dfrac{8x+x^2}{x^{\frac{1}{2}}} = 8x^{\frac{1}{2}}+x^{\frac{3}{2}}$

We now use the above rule to see that

$\int \dfrac{8x+x^2}{\sqrt{x}}\ dx$

$= \int 8x^{\frac{1}{2}}+x^{\frac{3}{2}}\ dx$

$= \frac{16}{3}x^{\frac{3}{2}}+\frac{2}{5}x^{\frac{5}{2}}+c$

Some more examples...

y	$\int y \, \mathrm{d}x$
$(x+2)(x+3) = x^2 + 5x + 6$	$\frac{1}{3}x^3 + \frac{5}{2}x^2 + 6x + c$
$\frac{x+3}{x^3} = x^{-2} + 3x^{-3}$	$-x^{-1} - \frac{3}{2}x^{-2} + c$
$\frac{2x+6}{\sqrt{x}} = \frac{2x+6}{x^{\frac{1}{2}}} = 2x^{\frac{1}{2}} + 6x^{-\frac{1}{2}}$	$\frac{4}{3}x^{\frac{3}{2}} + 12x^{\frac{1}{2}} + c$
$\frac{(x+3)(x+4)}{\sqrt{x}} = \frac{x^2+7x+12}{x^{\frac{1}{2}}} = x^{\frac{3}{2}} + 7x^{\frac{1}{2}} + 12x^{-\frac{1}{2}}$	$\frac{2}{5}x^{\frac{5}{2}} + \frac{14}{3}x^{\frac{3}{2}} + 24x^{\frac{1}{2}} + c$

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